Mechanics: Dynamics

FIZIKA SPhO Training

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1 Notes

In kinematics, we have discussed motion without caring about where it stems from. Now, we will go into details of how motion arises through forces.

1.1 Newton's Laws of Motion

Let's briefly recap Newton's Laws of Motion, which you should have learned long ago.

1.1.1 Newton's First Law

A body with no net force acting on it will move with either constant velocity or zero velocity, depending on the initial conditions. Such a body is said to be in equilibrium.

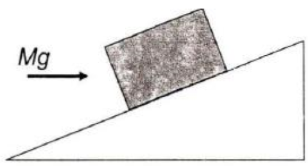
1.1.2 Newton's Second Law (N2L)

This should be second nature to all of you.

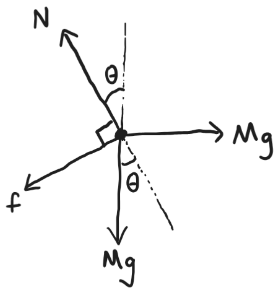
$$\mathbf{F} := m\mathbf{a} \tag{1}$$

You are highly recommended to draw free-body diagrams and label all forces in the set-up, so that you don't miss out any.

Example 1.1 (SPhO 2010)**.** A block of mass M rests on a fixed plane inclined at angle θ. A horizontal force of magnitude Mg is applied to the block, as shown in the figure below. The coefficient of static friction between the block and the plane is μ. (i) Assuming that the friction force between the block and the plane is large enough to keep the block at rest, determine the magnitude of the normal and friction forces (call then N and f) that the plane exerts on the block in terms of M and μ. (ii) Determine the **range** of angles θ for which the block remains at rest on the plane in terms of μ.



(i) Let's start by drawing a free-body diagram of the block.



You should realise it is better to resolve forces parallel to and perpendicular to the incline. Thus, applying the equilibrium condition along these two axes,

$$f = Mg\cos\theta - Mg\sin\theta$$

$$N = Mg\sin\theta + Mg\cos\theta$$

(ii) Two things could happen here.

1. The applied force is too small, and the block is on the verge of slipping downwards.
2. The applied force is too large, and the block is on the verge of slipping upwards.

It is a common mistake to miss out one of the cases! For either case, at the extreme case when the block is about to slip, we have $f = \mu N$.

For Case 1, the free-body diagram is similar to the one drawn above, except with friction pointing in the opposite direction. Thus, we have

$$f = Mg\sin\theta - Mg\cos\theta$$

instead, and N remains unchanged. Thus, at the extreme,

$$Mg\sin\theta - Mg\cos\theta = \mu\left(Mg\sin\theta + Mg\cos\theta\right)$$

$$\mu = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta} = \frac{\tan\theta - 1}{\tan\theta + 1} = \tan\left(\theta - \frac{\pi}{4}\right) \quad \Rightarrow \quad \theta = \frac{\pi}{4} + \tan^{-1}\mu$$

For Case 2, the free-body diagram is exactly the one drawn above. Thus, at the extreme,

$$Mg\cos\theta - Mg\sin\theta = \mu\left(Mg\sin\theta + Mg\cos\theta\right)$$

$$\mu = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} = \frac{1 - \tan\theta}{\tan\theta + 1} = \tan\left(\frac{\pi}{4} - \theta\right) \quad \Rightarrow \quad \theta = \frac{\pi}{4} - \tan^{-1}\mu$$

As these are the extreme cases, they give the bounds for our range of θ. Thus,

$$\frac{\pi}{4} - \tan^{-1}\mu \le \theta \le \frac{\pi}{4} + \tan^{-1}\mu$$

Again, before you conclude a problem, check that you haven't missed out any cases!

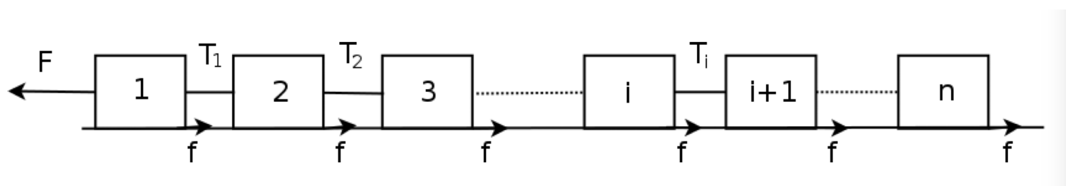
Unfortunately, most of the time, just using N2L is not enough to solve most mechanics questions. While forces are the most fundamental, you'll need more concepts which we will be introducing in the next sections, to solve Olympiad problems efficiently.

1.1.3 Choice of System

Depending on the situation, you are free to choose what objects define your system. You may then ignore the **internal forces** within your system.

The following example illustrates how to choose your system wisely.

Example 1.2. A force is applied on the left end of an arrangement of n identical blocks of mass m that are connected by strings. The coefficient of kinetic friction between the blocks and the ground is μ. All strings remain taut in the process of the blocks' motion. Find the tension T_i in the string between the ith and $(i+1)$th blocks as the blocks accelerate.



You might first consider your system to be all the blocks. By doing so, all the tensions are regarded as internal forces, which you can ignore. Thus, N2L is written as

$$F - nf = nMa$$

where $f = \mu N = \mu Mg$ is the kinetic friction on each block.

Instead, if we look at the first to the ith block as our system, N2L is written as

$$F - if - T_i = iMa$$

And, when we look at the ith to the nth block as our system, N2L is written as

$$T_i - (n-i)f = (n-i)Ma$$

We have 3 equations and 3 unknowns, so solving for T_i,

$$T_i = \frac{n-i}{n}F$$

1.1.4 Newton's Third Law

If body A exerts a force on body B, body B exerts a force of the same magnitude but opposite direction on body A. These two forces act on different bodies.

1.2 Systems of Masses

There are better ways to analyse dynamics than analysing $\mathbf{F} = m\mathbf{a}$ on every single particle.

1.2.1 Centre of Mass (CM)

Consider a system of N **discrete** point masses, $m_1, m_2, ..., m_N$, with positions $\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N$ respectively. The CM is located at

$$\mathbf{r}_{CM} = \frac{\sum_{i=1}^N m_i \mathbf{r}_i}{\sum_{i=1}^N m_i} = \frac{1}{M_{total}} \sum_{i=1}^N m_i \mathbf{r}_i \tag{2}$$

You may think of this as a "weighted sum" of masses, with the "weights" being their positions.

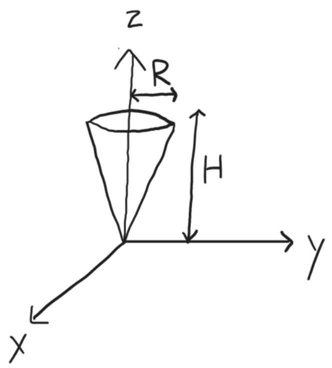
However, most real objects are not well-modelled by point masses. For a **continuous** mass distribution, the CM is located at

$$\mathbf{r}_{CM} = \frac{\int \mathbf{r}\, dm}{\int dm} = \frac{1}{M_{total}} \int \mathbf{r}\, dm \tag{3}$$

or in Cartesian coordinates, $x_{CM} = \frac{1}{M_{total}} \int x\, dm$, $y_{CM} = \frac{1}{M_{total}} \int y\, dm$, $z_{CM} = \frac{1}{M_{total}} \int z\, dm$.

In SPhO, you'll likely only find the CM for situations with symmetry along one or two axes. Using symmetry, you can quickly deduce the position of the CM along some axes. Here's an example to illustrate this.

Example 1.3. Consider a strange, **non-uniform** cone of base radius R and height H, with its tip at the origin. Its density varies along the z-axis as $\rho(z) = kz^3$ for some constant k. Find the position of its CM.



Notice that the non-uniformity only occurs along the z-axis. By symmetry of the cone through the z-axis, we can instantly conclude

$$x_{CM} = y_{CM} = 0$$

Hopefully you saw this instantly! That is the power of symmetry.

Back to the z-axis, consider disks of thickness dz and radius r at height z. By similar triangles, $r = \frac{R}{H}z$. Thus,

$$z_{CM} = \frac{\int z\,dm}{\int dm} = \frac{\int_0^H z(\rho\pi r^2\,dz)}{\int_0^H \rho\pi r^2\,dz} = \frac{\int_0^H k\pi\left(\frac{R}{H}\right)^2 z^6\,dz}{\int_0^H k\pi\left(\frac{R}{H}\right)^2 z^5\,dz} = \frac{\frac{H^7}{7}}{\frac{H^6}{6}} = \frac{6}{7}H$$

Thus, the CM is at $\left(0,\,0,\,\frac{6}{7}H\right)$.

The position of the CM is the point at which the total gravitational force *effectively* acts on the object, assuming a uniform gravitational field.

1.2.2 CM of Isolated Systems

From Equation (1), the CM of an isolated system ($\mathbf{F} = \mathbf{0}$) cannot accelerate. If the CM was initially at rest, then the CM will not move.

The *non-movement* of the CM is an essential idea. We call this the **conservation of CM.**

Example 1.4. Consider a boat of length L and mass M. A dog of mass m starts at the left end of the boat. The boat floats freely on water. After the dog runs to the right end and stops, how much has the boat moved from its initial position?

Because the boat floats freely, the boat–dog system experiences no net external force. Let the origin of our coordinate system be the initial position of the left end. Then,

$$x_{\text{CM},i} = \frac{M\left(\frac{L}{2}\right) + m\,(0)}{M+m} = \frac{ML}{2\,(M+m)}.$$

Suppose the boat has moved a distance x from its original position, where $x < 0$ if to the left and $x > 0$ if to the right. Then, the boat's center is now at $x + \frac{L}{2}$, and the dog is at $x + L$. Thus,

$$x_{\text{CM},f} = \frac{M\left(x + \frac{L}{2}\right) + m\,(x+L)}{M+m} = x + \frac{M\frac{L}{2} + mL}{M+m}.$$

By conservation of CM, $x_{\text{CM},i} = x_{\text{CM},f}$, so

$$x = \frac{ML}{2(M+m)} - \frac{M\frac{L}{2} + mL}{M+m} = -\frac{mL}{M+m},$$

with the negative sign meaning the boat moved to the left (opposite the dog's run).

In general, if you have a complex motion in between two well-defined states, this idea can be used. In this case, the complex motion is the dog's run, and the two well-defined states are the start and the end of the motion. Notice we didn't even specify the dog's velocity or how it runs, because it doesn't depend on it at all!

1.3 Momentum and Impulse

As seen earlier, the (linear) momentum $\mathbf{p}$ is defined by

$$\mathbf{p} = m\mathbf{v} \tag{4}$$

Impulse is defined by

$$\mathbf{J} = \int \mathbf{F}\, dt \tag{5}$$

The **impulse-momentum theorem** marries the two together:

$$\mathbf{J} = \Delta\mathbf{p} \tag{6}$$

1.3.1 Conservation of Momentum (COM)

In the absence of net external force, the **total momentum is conserved.** This is called the **conservation of momentum (COM).** Mathematically,

$$\mathbf{F} = \mathbf{0} \quad \Rightarrow \quad \mathbf{p} = \text{constant} \tag{7}$$

Or, if a net external force exists, then

$$\mathbf{F} := \frac{d\mathbf{p}}{dt} \tag{8}$$

This looks different from Equation (1). In reality, this definition of $\mathbf{F}$ is the correct one, because Equation (1) assumes a constant mass. You will see more examples on how to apply this to changing mass systems later on.

1.3.2 Collisions

In **all collisions,** the total momentum is conserved. In **perfectly elastic collisions,** the total kinetic energy is conserved too. In **perfectly inelastic collisions,** the objects stick together and have 0 relative velocity after collision.

Normal methods (which you should have learnt) of solving the equations conserving momentum along x and y-axes and conserving energy (for elastic collisions) always work. It may sometimes be more helpful to define the **coefficient of restitution (COR):**

$$\varepsilon = \frac{\text{relative speed of separation}}{\text{relative speed of approach}} = \frac{v_2 - v_1}{u_1 - u_2} \tag{9}$$

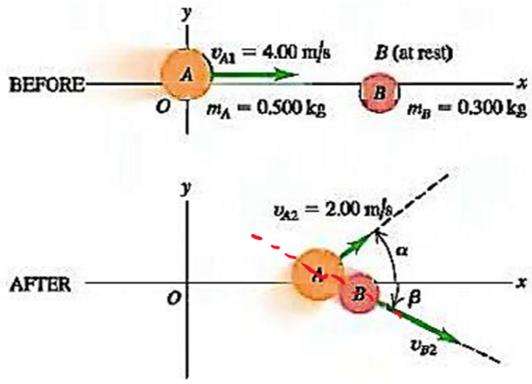
where u_1, u_2 are the initial speeds and v_1, v_2 are the final speeds. **Signs are important here,** since we are dealing with relative speeds!

The COR satisfies the following:

1. $0 \leq \varepsilon \leq 1$. For perfectly inelastic collisions, $\varepsilon = 0$. For perfectly elastic collisions, $\varepsilon = 1$.
2. It can only be applied **along the line of impact!**

What does "along the line of impact" mean? The following example illustrates.

Example 1.5. 2 pucks collide elastically on a frictionless table. Puck A has mass $m_A = 0.5$ kg and puck B has mass $m_B = 0.3$ kg. Puck A has an initial velocity of 4 m/s in the positive x-direction and a final velocity of 2 m/s in an unknown direction. Puck B is initially at rest. Determine the final speed v_{B2} of puck B and the angles α and β.



Of course, you can solve this like any normal collision problem. But, let's be neat and apply the fact that $\varepsilon = 1$ along the line of impact, for an elastic collision.

Conserving momentum perpendicular to the line of impact,

$$v_{A1} \sin\beta = v_{A2} \sin(\alpha + \beta)$$

Using $\varepsilon = 1$ parallel to the line of impact,

$$v_{A1} \cos\beta = v_{B2} - v_{A2} \cos(\alpha + \beta)$$

Not forgetting, conserving kinetic energy,

$$\frac{1}{2} m_A v_{A1}^2 = \frac{1}{2} m_A v_{A2}^2 + \frac{1}{2} m_B v_{B2}^2$$

This offers an advantage over conserving momentum in the x and y-axes. The equations (write them out yourself as an exercise!) are messier to deal with as they contain m_A and m_B, while our first 2 equations here don't.

1.3.3 Collisions in the CM Frame

The CM frame is a powerful tool to analyse collisions. We define the CM velocity:

$$\mathbf{v}_{CM} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} \tag{10}$$

In the CM frame, the total momentum of a system must be 0. Recall from kinematics how to transform quantities from one frame to another. To transform the velocity in the lab frame to the CM frame, we have

$$\mathbf{v}_{1,CM} = \mathbf{v}_{1,lab} - \mathbf{v}_{CM} \tag{11}$$

Now, transform the 2 masses that we are considering, m_1 and m_2, with lab frame velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively, into the CM frame. The new velocities are

$$\mathbf{v}_{1,CM} = \mathbf{v}_1 - \mathbf{v}_{CM} = \frac{m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2) \tag{12}$$

$$\mathbf{v}_{2,CM} = \mathbf{v}_2 - \mathbf{v}_{CM} = \frac{m_1}{m_1 + m_2}(\mathbf{v}_2 - \mathbf{v}_1) \tag{13}$$

You may verify yourself that $m_1\mathbf{v}_{1,CM} + m_2\mathbf{v}_{2,CM} = \mathbf{0}$. Thus, the outgoing velocities after collision (denoted as primed quantities) in the CM frame, by COM, satisfy

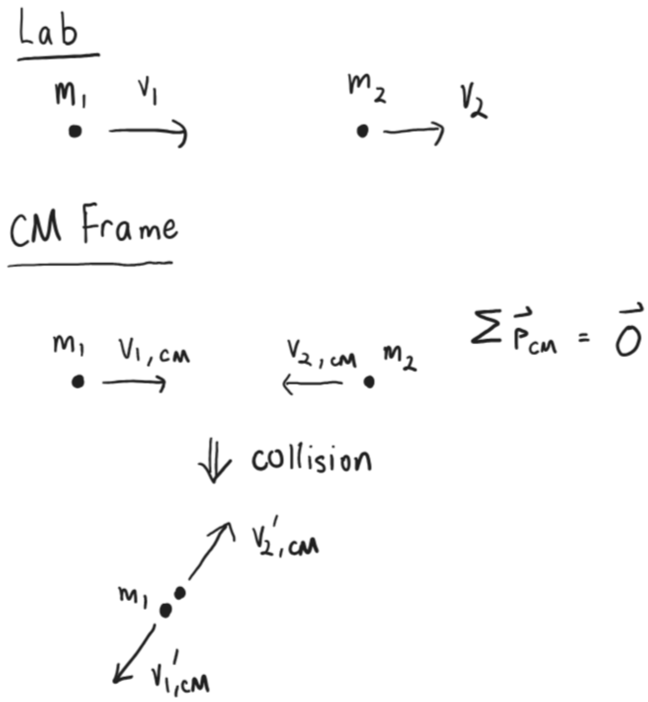
$$m_1\mathbf{v}'_{1,CM} + m_2\mathbf{v}'_{2,CM} = \mathbf{0} \tag{14}$$

This means

$$\mathbf{v}'_{2,CM} = -\frac{m_1}{m_2}\mathbf{v}'_{1,CM} \tag{15}$$

which means $\mathbf{v}'_{2,CM}$ and $\mathbf{v}'_{1,CM}$ point in opposite directions.

Thus, this means that in the CM frame, the masses always leave in directions opposite to each other! The figure below illustrates what is happening.



Remark. If you use the CM frame to find some quantity, be careful and remember to convert it back to the lab frame!

Now, let's see how the CM frame can be helpful in solving collision problems.

Example 1.6. Find the maximum angle of deviation, θ, that a particle of mass m_1 can deviate from its original path after an elastic collision with mass m_2 initially at rest.

Let m_1 initially move with velocity $\mathbf{v}_1$, so that

$$\mathbf{v}_{CM} = \frac{m_1}{m_1 + m_2}\mathbf{v}_1$$

The initial velocities of the particles in the CM frame are

$$\mathbf{v}_{1,CM} = \mathbf{v}_1 - \frac{m_1}{m_1+m_2}\mathbf{v}_1 = \frac{m_2}{m_1+m_2}\mathbf{v}_1$$
$$\mathbf{v}_{2,CM} = -\frac{m_1}{m_2}\mathbf{v}_{1,CM} = -\frac{m_1}{m_1+m_2}\mathbf{v}_1$$

You should verify that the total momentum is 0.

After the collision, $\mathbf{v}'_{1,CM}$ can be in any direction - we don't know yet. Regardless of the direction, in the lab frame, the final velocity will be $\mathbf{v}'_1 = \mathbf{v}_{CM} + \mathbf{v}'_{1,CM}$.

However, we know that $\left|\mathbf{v}'_{1,CM}\right|$ is fixed, regardless of direction. Geometrically, if we fix an origin for the vector $\mathbf{v}'_{1,CM}$, the locus of all points it can point to is a circle of radius $\left|\mathbf{v}'_{1,CM}\right|$.

The last piece of information we need is that the collision is elastic. As energy is conserved,

$$\frac{1}{2}m_1\left|\mathbf{v}_{1,CM}\right|^2 + \frac{1}{2}m_2\left|\mathbf{v}_{2,CM}\right|^2 = \frac{1}{2}m_1\left|\mathbf{v}'_{1,CM}\right|^2 + \frac{1}{2}m_2\left|\mathbf{v}'_{2,CM}\right|^2$$

And, from Equation (15),

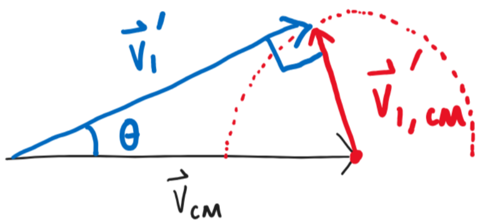
$$\mathbf{v}'_{2,CM} = -\frac{m_1}{m_2}\mathbf{v}'_{1,CM} \quad \Rightarrow \quad \left|\mathbf{v}'_{2,CM}\right| = \frac{m_1}{m_2}\left|\mathbf{v}'_{1,CM}\right|$$

Substituting this into the COE equation, you will obtain

$$\left|\mathbf{v}'_{1,CM}\right| = \left|\mathbf{v}_{1,CM}\right| = \frac{m_2}{m_1+m_2}\left|\mathbf{v}_1\right|$$

which means that the *speed* of m_1 remains unchanged after an elastic collision, in the CM frame!

Thus, we may draw a vector diagram illustrating $\mathbf{v}'_1 = \mathbf{v}_{CM} + \mathbf{v}'_{1,CM}$ as such:



Notice that θ is maximised when $\mathbf{v}'_1$ is tangent to the semi-circle, containing all possible directions $\mathbf{v}'_{1,CM}$ could point in relative to $\mathbf{v}_{CM}$. Thus,

$$\theta = \sin^{-1}\left(\frac{\left|\mathbf{v}'_{1,CM}\right|}{\left|\mathbf{v}_{CM}\right|}\right) = \sin^{-1}\left(\frac{\left|\mathbf{v}_{1,CM}\right|}{\left|\mathbf{v}_{CM}\right|}\right) = \sin^{-1}\left(\frac{m_2}{m_1}\right)$$

This doesn't make sense when $m_2 > m_1$. But physically, we know that m_1 still has to bounce back somehow. In this case, it will just bounce back directly, giving $\theta = \pi$. Thus, you should write your final answer as:

$$\theta = \begin{cases} \arcsin\left(\frac{m_2}{m_1}\right), & m_2 < m_1 \\ \pi, & m_2 > m_1 \end{cases}$$

So, before you finish a problem, check it for extreme cases and consider all possible regimes!

Of course, you could have done this problem without going into the CM frame at all. You may try this as an exercise. By doing so, you would have to grind through a lot of algebra, which could have been avoided if you knew about the CM frame method presented here!

1.4 Work, Energy and Power

Generally speaking, there are two types of energy you need to know.

1. **Kinetic Energy (KE):** It is simply given by $\frac{1}{2}mv^2$.
2. **Potential Energy (PE):** There are many forms of PE. Most commonly, you will encounter gravitational PE (GPE), elastic PE (EPE), and electric PE (also EPE).

You should also know that **power** is the rate of doing work,

$$P_{instantaneous} = \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v}, \ P_{average} = \frac{\Delta W}{\Delta t} = \frac{\int P\, dt}{\int dt} \tag{16}$$

1.4.1 Conservation of Energy (COE)

The total energy of an **isolated system** remains constant. Loosely speaking, we can write

$$K_i + U_i = K_f + U_f \tag{17}$$

accounting for both kinetic (K) and potential (U) energies.

1.4.2 Work Done

The work done by a force $\mathbf{F}$ between two points, $\mathbf{r}_i$ and $\mathbf{r}_f$ is defined by

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} \tag{18}$$

In Cartesian coordinates, $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$.

Example 1.7 (Ricardo)**.** Find the work done by the force $\mathbf{F} = x^2\hat{\mathbf{i}} - 2xy^2\hat{\mathbf{j}}$ between $(0,0)$ and $(2,4)$, through a curve given by $y = x^2$.

In 2D, we have $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$, thus

$$\mathbf{F} \cdot d\mathbf{r} = \left(x^2\hat{\mathbf{i}} - 2xy^2\hat{\mathbf{j}}\right) \cdot \left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}\right) = x^2\, dx - 2xy^2\, dy$$

Since $y = x^2$, $dy = 2x\, dx$, thus

$$\mathbf{F} \cdot d\mathbf{r} = x^2\, dx - 2x\left(x^2\right)^2 (2x\, dx) = \left(x^2 - 4x^6\right) dx$$

Thus, the work done is

$$W = \int_0^2 \left(x^2 - 4x^6\right) dx = \left[\frac{x^3}{3} - \frac{4x^7}{7}\right]_0^2 = -\frac{1480}{21}$$

Work is linked to kinetic energy through the **work-energy theorem:**

$$W_{total} = \Delta K \tag{19}$$

1.4.3 Conservative Forces

Conservative forces are a special type of forces whereby the work done in moving a particle between two points is **independent of the path taken,** and only depends on the initial and final positions.

Only for conservative forces, we may ascribe a **potential energy function,** U, whereby

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}, \ W_{A\to B} = \int_A^B \mathbf{F} \cdot d\mathbf{r} := U_A - U_B \tag{20}$$

where $\mathbf{r}_0$ is a reference point (usually at ∞).

For a conservative force, the following results hold:

1. $W = -\Delta U$. **The negative sign is important!**
2. $W_{closed\,loop} = 0$. The work done by a conservative force through a closed loop is 0.

Some examples of **conservative forces** are gravitational force and spring force. These forces are conservative because they have potential energy functions - the GPE and EPE respectively!

In contrast, some examples of **non-conservative forces** are friction and air resistance. Clearly, the work done by these forces depends on the path taken by the particle.

Revisiting our COE equation in Equation (17), work done gives us an alternative way of writing it, in terms of the work done by non-conservative forces,

$$\Delta E = W_{non-conservative} \tag{21}$$

where $E = K + U$ is the total energy. This means that **total energy is conserved if there is no net work done by non-conservative forces.** You already know this - when friction is present, you cannot apply COE, because the non-conservative friction force does work!

Example 1.8. Given a charge q at the origin and charge Q at $\mathbf{r}$, find the electric potential energy. The force acting on Q as a function of position is $\mathbf{F} = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$. Take the point at infinity to have 0 potential energy.

Using Equation (20), we have

$$U(r) = -\int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = -\int_{\infty}^{r} \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, dr = -\int_{\infty}^{r} \frac{Qq}{4\pi\varepsilon_0 r^2}\, dr = \left[\frac{Qq}{4\pi\varepsilon_0 r}\right]_{\infty}^{r} = \frac{Qq}{4\pi\varepsilon_0 r}$$

The lower bound of integration here is ∞, since that is our reference point of 0 potential energy.

1.4.4 Energy With Multiple Particles

There is a subtlety when you are dealing with the total energy of a system with multiple particles. The confusion arises in the **potential energy.**

Example 1.9. Two masses m_1 and m_2 are currently traveling at speeds v_1 and v_2 respectively. If they are connected by a spring with a zero relaxed length that is currently of length x, determine the total mechanical energy of the system of the two masses. Neglect friction and other dissipative forces.

You may be tempted to write down the total energy of *each* mass, and then sum them together. By doing so,

$$E_{1,WRONG} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}kx^2,\ E_{2,WRONG} = \frac{1}{2}m_2 v_2^2 + \frac{1}{2}kx^2$$

$$E_{tot,WRONG} = E_{1,WRONG} + E_{2,WRONG} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + kx^2$$

However, the issue with this is that **potential energy doesn't belong to any mass!** Potential energy belongs to the *interaction* between the masses. By doing this, you double-counted the potential energy.

The correct answer only counts the potential energy interaction once, and is

$$E_{tot} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}kx^2$$

To avoid double-counting potential energy, you should always look at the **number of pairwise interactions** (between 2 particles) involved, instead of the number of particles!

1.4.5 Partial Derivatives, Forces, and Energy

We may relate a conservative force and its potential energy as such:

$$F_x = -\frac{\partial U}{\partial x},\ F_y = -\frac{\partial U}{\partial y},\ F_z = -\frac{\partial U}{\partial z} \tag{22}$$

where the ∂ symbols denote partial derivatives (i.e. differentiating U with respective to the stated coordinate, while keeping all other variables constant).

Partial derivatives provide us a way to check the **conservativeness of a force.** A force (in 3D) is conservative if and only if

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x},\ \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x},\ \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \tag{23}$$

The reason why this condition guarantees conservativeness is purely mathematical and requires vector calculus. You do not need to know the reason, you just need to know this condition.

1.4.6 Stability Analysis

Consider a 1D system in equilibrium. If **only conservative forces exist** in the system, then $F = -\frac{dU}{dx}$. Thus, the condition for equilibrium is

$$\frac{dU}{dx} = 0 \tag{24}$$

We can classify equilibria into three types:

Stable Equilibrium: The body tends to return to its original position after being disturbed. The force that brings the body back to its original position is called a **restoring force.** (More on this will be covered in oscillations.)

The condition for stable equilibrium is

$$\frac{d^2U}{dx^2} > 0 \tag{25}$$

Unstable Equilibrium: Opposite to stable equilibrium, the body tends to move further away from its original position after being disturbed.

The condition for unstable equilibrium is

$$\frac{d^2U}{dx^2} < 0 \tag{26}$$

Neutral Equilibrium: In this case, the body neither tends to return to its origin position nor depart more widely from it.

The condition for neutral equilibrium is

$$\frac{d^2U}{dx^2} = 0,\ \frac{d^3U}{dx^3} = 0,\ \text{and so on...} \tag{27}$$

Note that **all higher derivatives must be 0!** Only checking the second derivative is insufficient. Thankfully, neutral equilibrium is rare and you'll basically never encounter it.

1.5 Ideas

Many tricky mechanics problems involve the use of the following ideas.

1.5.1 Energy and Frames of Reference

No matter what frame of reference you choose, total energy is always conserved! However, it is quintessential to choose the right frame of reference. **Even though total energy is conserved, kinetic energies and changes in kinetic energy can differ between frames.**

The following example illustrates a subtlety with energy between frames.

Example 1.10 (Kevin Zhou)**.** Alice steps on the gas pedal on her car. Bob, who is standing on the sidewalk, sees Alice's car accelerate from rest to 10 mph. Charlie, who is passing by in another car, sees Alice's car accelerate from 10 mph to 20 mph. Hence Charlie sees the kinetic energy of Alice's car increase by three times as much. How is this compatible with energy conservation, given that the same amount of gas was burned in both frames?

The difference arises due to the change in kinetic energy **of the Earth.** (Don't forget it exists!)

In Bob's frame, the final kinetic energies of the Earth and the car are

$$K_{Earth,B} = \frac{p^2}{2M},\ K_{car,B} = \frac{p^2}{2m}$$

where p is the total impulse given to the car (recall that an impulse is a change in momentum). As the Earth's mass M is massive, the former is negligible. As the car's mass m is much smaller than M, the latter is the only important quantity. This means the gain in kinetic energy of the Earth in Bob's frame is negligible.

However, this reasoning doesn't apply in Charlie's frame. In Charlie's frame, he sees Earth having some initial momentum P, as the Earth is not stationary with respect to himself. Thus, the change in kinetic energy of the Earth is

$$\Delta K_{Earth,C} = \frac{(P-p)^2 - P^2}{2M} = -\frac{Pp}{M} + \frac{p^2}{2M}$$

Again, the second term is negligible. However, the first term isn't, as it is only linear in p!

Thus, the *decrease* in Earth's kinetic energy accounts for the *increase* in the car's kinetic energy in Charlie's frame.

The key lesson from this example is that when we want to find how energy changes in a small object interacting with a larger object, we should almost always **go into the frame of the larger object.** You will need to use this crucial idea later!

1.5.2 Extended Systems

These types of problems are most likely too difficult to come out in a modern SPhO paper, but nonetheless is very important to learn.

We will be using infinitesimal analysis extensively to analyse how a system changes over a small amount of time, dt. This method of analysis is especially useful to analyse systems that are changing in mass. Specifically, we will introduce the impulse-momentum approach.

First, let us look at an example to show why $\mathbf{F} = m\mathbf{a}$ might not work well.

Example 1.11 (Morin)**.** Consider a cart that has some initial mass and is moving at speed v. We drop sand vertically into the cart at a rate of σ (dimensions of mass per unit time). The sand has no horizontal speed relative to the cart.. Find the external force needed to maintain the speed of the cart at v.

If we just blindly apply N2L as per Equation (1), we will find that

$$F = ma = (m)(0) = 0$$

since the cart is moving at a constant speed. This means that no force is required to maintain the cart at constant speed v, even though its mass is constantly increasing. However, this is clearly wrong. Imagine that after a long time, the cart becomes massive - then surely, you would need to apply an external force to keep it going!

The correct way is to consider the cart + the sand on the cart, with total mass m. Note that m changes with time. If the mass of the empty cart is m_0, then

$$m(t) = m_0 + \sigma t \quad \Rightarrow \quad p(t) = (m_0 + \sigma t)\, v$$

Using N2L as per Equation (8),

$$F = \frac{dp}{dt} = \frac{d}{dt}\left((m_0 + \sigma t)\, v\right) = \sigma v$$

Another correct way is to utilise the impulse-momentum theorem,

$$\Delta p = \int F_{tot}\, dt$$

which is equivalent to saying

$$\Delta p = \sum_i F_i\, \Delta t$$

This might look similar to $\frac{dp}{dt}$, but it is actually not the same when we do questions. The previous must be applied to the whole system, whereby here we can define a sub-system, that still obeys the fact that a system is defined to have the same mass in time interval t to $t + \Delta t$ in the time of our analysis.

In this case, we can define our sub-system as the mass in the cart and an infinitesimal mass dm that lands in the cart.

$$\begin{aligned} dp &= v\, dm \\ \frac{dp}{dt} &= v\frac{dm}{dt} \\ F &= \sigma v \end{aligned}$$

Alternatively, to solve this problem, you might have considered this expansion

$$F = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt} \tag{28}$$

From this equation, we see that $F = \sigma v$ as well.

However, this is **not exactly correct,** as our system is not the same at time t and time $t + dt$ when we have performed our differentiation. Our system at $t + dt$ contains $\sigma\, dt$ more sand than the system before that; however, our system cannot change throughout our analysis!

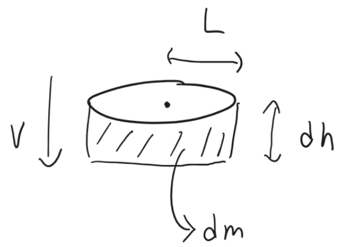
From the above analysis, using the impulse-momentum theorem should be more intuitive to you. It may be physically weird to perform calculations on the entire system, even when some part of it is irrelevant, as we have done using $\frac{dp}{dt}$. Instead, we can nicely define a sub-system of our interest and perform impulse-momentum directly on that sub-system.

Another reason why impulse-momentum is preferred is that $\frac{dp}{dt}$ is evaluated at a point. However, our system is evaluated over a period of time, possibly non-infinitesimal. Hence, it would be false to use $\frac{dp}{dt}$ as it is restricted to infinitesimal analysis, while $\Delta p = F\Delta t$ still works as we didn't need to evaluate it at just one point.

Let's look at another more involved example, showing the power of infinitesimal analysis.

Example 1.12. A helicopter of mass M has rotor blades of length L. It has a net downward acceleration of $a < g$. What (constant) speed is it propelling air of density ρ downwards?

In light of the method above, we define our system to be the helicopter + a small cylinder of air, with infinitesimal thickness dh above the blade. We consider an instant in time at t. This cylinder of air will be sent downwards by the blade at a time dt later.



Let's define the speed of the air to be $v = \frac{dh}{dt}$ and the speed of the helicopter to be w. Applying the impulse-momentum theorem,

$$p_f - p_i = F\,dt$$

Here, the net force acting on our defined system is just the total gravitational force,

$$F = \left(M + \rho\pi L^2\,dh\right) g$$

The initial momentum is due to only the moving helicopter, since the cylinder of air is initially stationary before it is propelled by the blades.

$$p_i = Mw$$

The final momentum is due to both the moving helicopter, which acquires a new speed $w + dw$, and the moving cylinder of air, which acquires a speed v.

$$p_f = M(w + dw) + \rho\pi L^2 v\,dh$$

All in all, we can write

$$M(w + dw) + \rho\pi L^2 v\,dh - Mw = (M + \rho\pi L^2\,dh)g\,dt = Mg\,dt + \rho\pi L^2 g(dh)(dt) = Mg\,dt$$

where we have neglected the $\rho\pi L^2 g(dh)(dt)$ term, as it is second order in infinitesimal quantities.

Thus, rearranging,

$$M\,dw + \rho\pi L^2 v\,dh = Mg\,dt \quad \Rightarrow \quad M\frac{dw}{dt} + \rho\pi L^2 v\frac{dh}{dt} = Mg$$

Remembering that $\frac{dw}{dt} = a$ and $\frac{dh}{dt} = v$, thus

$$Ma + \rho\pi L^2 v^2 = Mg$$

$$v = \sqrt{\frac{M(g-a)}{\rho\pi L^2}}$$

Remark. You may think it is not physical that the blade can push the cylinder of air from a speed of 0 to v in time dt, as this would require infinite acceleration. However, take note that the mass is also infinitesimal! So, the net effect is a finite force, which makes sense.

1.5.3 Pulley Problems

One last type of problem you need to know is the pulley problem. Here, you'll have to take note of two things.

1. **All pulleys are usually assumed to be massless.** This means that they must have **no net force,** otherwise they would experience an infinite acceleration. However that *doesn't* mean they must have no acceleration! They can still experience a finite acceleration, yet have no net force, since $m = 0$.

2. **All strings are usually assumed to be light and inextensible.** This means that they have no mass, and cannot be stretched (i.e. have a constant length). The latter is usually called the **conservation of string.**

In particular, the conservation of string is very powerful. By knowing how to apply it, you will be able to get an additional equation, relating all the relevant accelerations.

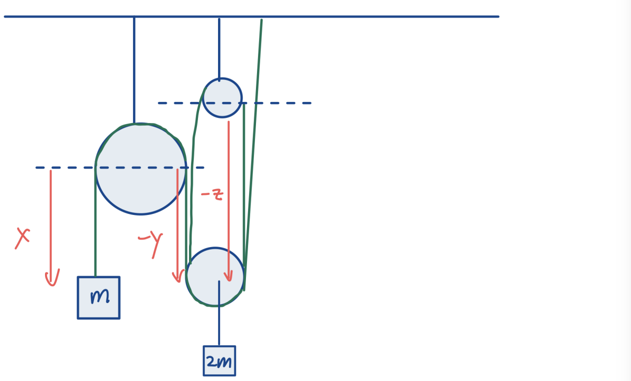
A recommended guideline to follow to apply conservation of string is as such:

1. Draw datum lines on all the fixed pulleys and relevant moving pulleys.

2. The sum of all lengths on one piece of string is constant. This means that the derivative with respect to time is zero.

3. When defining length, each new segment or piece of string should have a different index (x,y,z). The sign of these lengths depends on whether the string lengthens or shortens (you have to imagine). We define positive as lengthening, and negative as shortening.

Example 1.13. Relate the accelerations of the block with mass m and block with mass $2m$.



Here, we defined the coordinate of the left mass as x, the length of the middle string as $-y$ and the rest of the three pieces wrapping around the two pulleys as $-z$.

It is easy to see that $-\ddot{y} = -\ddot{z}$ because both pulleys on top are connected to the ceiling (and are hence immovable).

Then, by point 2, we have

$$x - y - 3z = \text{constant}$$

and thus, taking the derivative on both sides with respect to time,

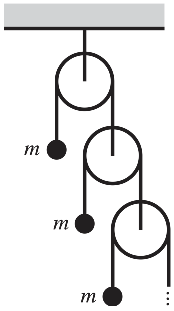
$$\ddot{x} - \ddot{y} - 3\ddot{z} = 0$$

We know that $a_1 = \ddot{x}$ and $a_2 = -\ddot{y} = -\ddot{z}$. Thus, we have $a_1 = -4a_2$.

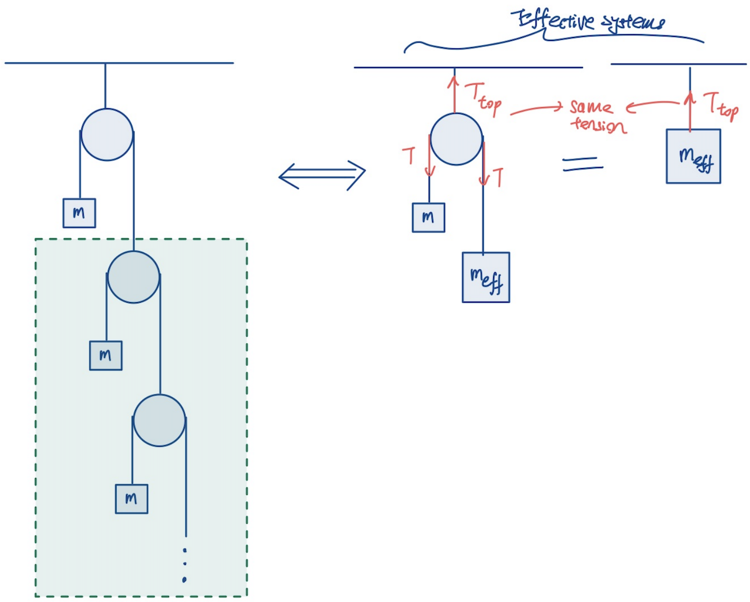
1.5.4 Perspective

This idea deals with infinitely recurring systems. Of course, we will be tackling the notorious **infinite pulley!**

Example 1.14. Consider the infinite pulley set-up below. Find the acceleration of the left-most mass if everything were initially held at rest and then released.



The key to solving this kind of problems is to recognise the **correct perspective** to go into - what characterises the system. In this case, we should go into the perspective of the ceiling, and find an effective mass that generates the same amount of tension on the first rope as if the whole system is replaced by the same effective mass! Here is an illustration.



From this diagram, it's clear that when we go into the perspective of the ceiling, we can treat the infinite pulley as effectively a simple pulley with mass m on one side, and an effective mass m_{eff} on the other. The m_{eff} replaces the rest of the pulley system, which, due to its infinitely recurring nature, is the same as the whole system itself!

This set-up has to produce the exact same tension as if just an effective mass were to be hanged. Using this idea, let's do the problem. The tension in the top string is given by

$$T_{top} = 2T$$

since there must be no net force on the massless pulley.

Now, considering the simple pulley consisting of m and m_{eff} (clearly $m_{eff} > m$), N2L gives

$$m_{eff}g - T = m_{eff}a,\ T - mg = ma \quad \Rightarrow \quad T = \frac{2mm_{eff}}{m + m_{eff}}g \quad \Rightarrow \quad T_{top} = \frac{4mm_{eff}}{m + m_{eff}}g$$

However, at the same time, referring to the rightmost diagram of the pulley,

$$T_{top} = m_{eff}g$$

Equating these two expressions, we have

$$m_{eff}g = \frac{4mm_{eff}}{m + m_{eff}}g \quad \Rightarrow \quad m_{eff} = 3m$$

Thus, the acceleration of the leftmost mass m is given by

$$a = \frac{3m - m}{3m + m}g = \frac{g}{2}$$

Such a thought process (called "self-similarity" in some books) can be used in many recurring systems, as long as you can identify the correct **constant parameter** after you have replaced the 2nd repeat unit (and onwards) by an effective unit. The rest is just algebra.

1.6 References

An Introduction to Mechanics by *Kleppner & Kolenkow*

Mechanics Handout by *Jaan Kalda*

Competitive Physics by *Wang & Ricardo*

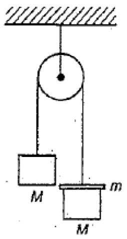
University Physics by *Young & Freedman*

An Introduction to Classical Mechanics by *David Morin*

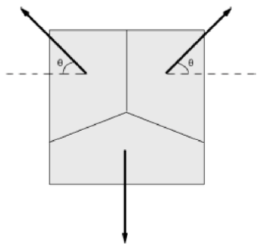
2 Problems

Problems are arranged in roughly increasing difficulty.

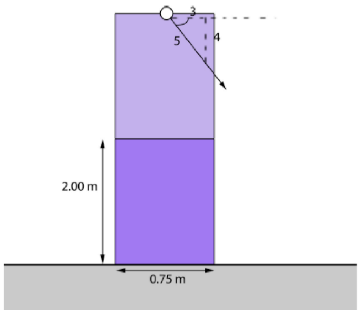
Problem 2.1 (SPhO 2016)**.** An experiment is performed to determine the value of the gravitational acceleration g on Earth. Two equal masses of mass M hang at rest from the ends of a string on each side of a pulley (see figure below). A mass $m = 0.01M$ is placed on the mass of the right. After the heavier side has moved down from rest by $h = 1\,\text{m}$, the small mass m is removed. The system continues to move for the next $1\,\text{s}$, covering an additional distance $H = 0.312\,\text{m}$. Find the value of g.



Problem 2.2 (SPhO 2007)**.** A rocket is projected upwards and explodes into three equally massive fragments, just as it reaches the top of its flight. One of the fragments is observed to fall downwards and land on the ground in time t_1, while the other two land at a time t_2 after the burst. Determine the height h, in terms of t_1 and t_2, at which the fragmentation occurs.



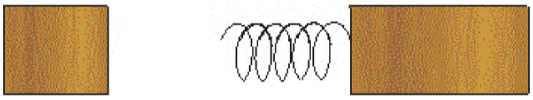
Problem 2.3 (SPhO 2007)**.** A column is formed from two marble blocks each weighing $15\,\text{kN}$ sitting atop each other. A rope to which a force F is applied is attached to the top (of the top marble block). The friction coefficient between the marble and ground is $\mu_{MG} = 0.15$ while the friction coefficient between the marble blocks is $\mu_{MM} = 0.20$. What is the maximum load F_{max} that can be applied, given that the force is applied in the direction in the figure below?



Problem 2.4 (SPhO 2012)**.** A platform scale is calibrated to indicate the mass (in kg) of an object placed on it. Particles fall from a height of $3.5\,\text{m}$ and collide with the balance pan of the scale. Assuming that the collisions are elastic (the particles rebound upwards with the same speed they had before hitting the pan), determine the *average* scale reading if each particle has mass $110\,\text{g}$ and collisions occur at a rate of $42\,\text{s}^{-1}$.

Problem 2.5 (Morin)**.** A tennis ball of mass m_2 sits on top of a basketball with mass $m_1 >> m_2$. The bottom of the basketball is at height h above the ground. After the balls are dropped, what is the maximum height the tennis ball bounces to? All collisions are elastic. Watch this demonstration for more fun! (You probably worked this out purely in the lab frame. Is there *another frame* you can go into for a more elegant solution?)

Problem 2.6 (SPhO 2002)**.** A block P of mass m slides along a horizontal frictionless track with speed v. The block collides with another stationary block Q of mass M, that has a light spring of spring constant k attached to it. Find the maximum compression of the spring, stating your assumptions.



Problem 2.7 (USAPhO 2019)**.** Two blocks, A and B, of the same mass are on a fixed inclined plane, which makes a 30° angle with the horizontal. At time $t = 0$, A is a distance $l = 5\,\text{cm}$ along the incline above B, and both blocks are at rest. Suppose the coefficients of friction (assume same for static and kinetic) between the blocks and the incline are $\mu_A = \frac{\sqrt{3}}{6}$ and $\mu_B = \frac{\sqrt{3}}{3}$, and that the blocks collide perfectly elastically. Let $v_A(t)$ and $v_B(t)$ be the speeds of the blocks down the incline. Use $g = 10\,\text{m/s}^2$, assume both blocks stay on the incline for the entire time, and neglect the size of the blocks. (i) Graph $v_A(t)$ and $v_B(t)$ from $t = 0\,\text{s}$ to $t = 1\,\text{s}$ with a solid and dashed line respectively on the same graph. Mark the times at which collisions occur. (ii) Derive an expression for the total distance block A has moved from its original position right after its nth collision, in terms of l and n. (iii) From now on, suppose $\mu_B = \frac{\sqrt{3}}{2}$ instead, while μ_A remains the same. On another graph, graph $v_A(t)$ and $v_B(t)$ from $t = 0\,\text{s}$ to $t = 1\,\text{s}$ with a solid and dashed line respectively. Mark the times at which collisions occur. (iv) At time $t = 1\,\text{s}$, how far has block A moved from its original position?

Problem 2.8 (Ricardo)**.** N identical men, with **total mass** m, stand on top of a cart of mass M initially at rest. When a man jumps off, he jumps with velocity u *relative* to the cart, in the opposite direction of the cart's motion. (i) If all the men jump off *at the same time,* find the final speed of the cart. (ii) If the men jump off *one at a time,* find the final speed of the cart. You may leave your answer in terms of a sum.

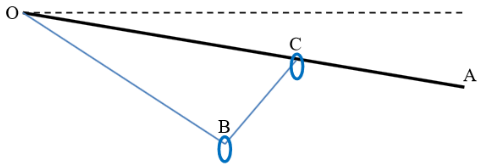
Problem 2.9. A rocket of initial total mass M starting at rest gains speed by ejecting fuel. Fuel is ejected opposite to the rocket's direction of motion at a uniform mass rate b (mass per unit time), at a speed u *relative* to the rocket. (i) Find the speed of the rocket as a function of time, if the rocket travels in zero gravity. (ii) Find the speed of the rocket as a function of time, if the rocket travels vertically upwards against a constant gravitational field g. (Hint: Consider infinitesimal analysis.)

Remark. After you have done this and the previous problem, you should realise that these problems are equivalent if we take $N \to \infty$!

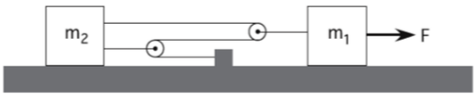
Problem 2.10 (SPhO 2003)**.** A nucleus of mass M_1 and kinetic energy K_1 moves along the x-axis and is incident on a stationary nucleus of mass M_2. The collision results in two new nuclei, one of them of mass M_3 and kinetic energy K_3, moving at an angle of ϕ to the x-axis and the other one of mass M_4 and kinetic energy K_4, moving at an angle of θ to the x-axis. In the non-relativistic case, show that the Q-value (defined as the difference in energies before and after the reaction) is

$$Q = K_3\left(1 + \frac{M_3}{M_4}\right) - K_1\left(1 - \frac{M_1}{M_4}\right) - \frac{2\sqrt{M_1 K_1 M_3 K_3}}{M_4}\cos\phi$$

Problem 2.11 (Ricardo)**.** A thin rough rod OA is fixed at an angle of 30° to the horizontal. A light inextensible string of length l has one end fixed at O, passes through a smooth ring B of mass $2m$, and is attached at its other end to a small ring C of mass m which slides on the rod. The coefficient of friction between the rod and ring C is $\frac{1}{4}$. If the system is in equilibrium, find the least possible distance between O and C.



Problem 2.12 (Ricardo)**.** Two blocks 1 and 2 rest on a frictionless horizontal surface. They are connected by three massless strings and two frictionless, massless pulleys as shown above. A force F is applied to block 1. What is the resulting acceleration of block 1?



Problem 2.13. Two masses of $m_1 > m_2$ are connected by a string, and the string is placed over a rod of radius R, essentially forming a pulley. The coefficient of friction (assume same for static and kinetic) between the string and the rod is μ. The rod is fixed and cannot rotate. What is the largest ratio $\frac{m_1}{m_2}$ before the string starts to slide? This set-up is linked to the Capstan equation, used by sailors to anchor their ships effectively. (Hint: Consider the forces acting on an infinitesimal piece of string, and balance them.)

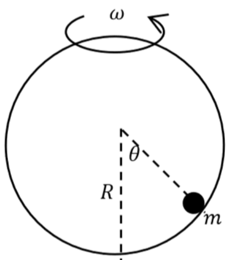
Problem 2.14. A *perfectly flexible* uniform chain of mass M and length L is suspended vertically with its lowest end barely touching the ground. It is released and crumples onto the ground. Determine the normal force as a function of time. (Hint: Each link of a *perfectly flexible* chain crumples when it hits the ground, without pulling the rest of the chain downward with it.)

Problem 2.15 (SPhO 2006)**.** A raindrop is falling through a cloud of water droplets. It gains mass as it falls, as the water droplets adhere to it. If the mass of the raindrop is proportional to the distance x it travels in the cloud, show that the equation of motion is

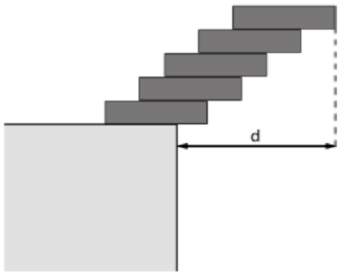
$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 = gx$$

Given that the proportionality constant for the mass is $k = 2$ g/m, find the mass of the raindrop at $t = 0.5$ s. (Hint: Consider a smart ansatz!)

Problem 2.16 (Ricardo)**.** A small bead is put on a frictionless hoop which is rotating with angular speed ω. The bead rotates with the hoop with the same angular speed. Let θ be defined as per the diagram below. (i) Using **forces,** determine the value(s) of θ that correspond to equilibrium. (ii) Using **potential energy,** determine the value(s) of θ that correspond to equilibrium. (iii) Determine the type(s) of equilibrium for the equilibrium angle(s) obtained.



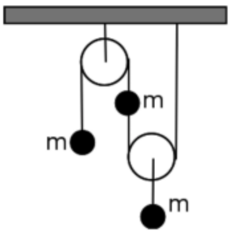
Problem 2.17 (SPhO 2007)**.** You have 5 blocks of dimension 1 m by 1 m and 25 mm thick. What is the furthest overhang from the edge of a table before they topple? (You can arrange the blocks in *any* orientation; the figure below is slightly misleading. The longer length of the blocks in the figure below *might not* be 1 m!)



Problem 2.18. A smooth wedge, whose central cross–section is a triangle ABC, right-angled at C, rests with the face containing AB on a smooth horizontal plane. When the wedge is held fixed, a particle released from rest takes a time t_1 to slide the full length of CA. The corresponding time for CB is t_2. Show that $\tan A = \dfrac{t_2}{t_1}$, and find AB in terms of t_1 and t_2.
If the mass of the wedge is n times that of the particle and the wedge is free to move, show that the time of sliding down CA becomes

$$t_1\sqrt{1 - \frac{t_2^2}{(n+1)(t_1^2+t_2^2)}}.$$

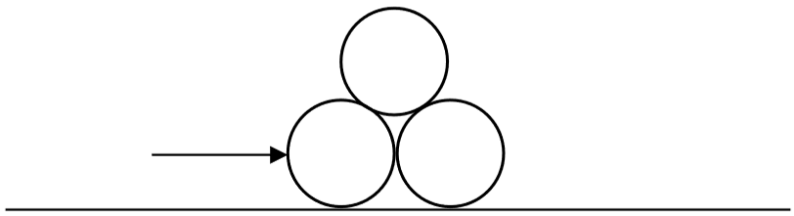
Problem 2.19. Determine the acceleration of all the masses in the setup below.



3 Advanced Problems

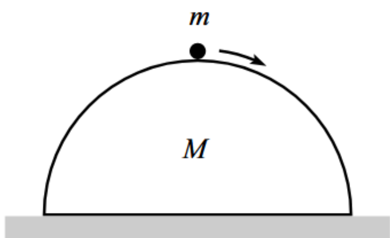
These problems are way too difficult to be tested in a modern-day SPhO. If you have completed all the previous problems and are down for a challenge, try these!

Problem 3.1 (Indonesia)**.** Three identical cylinders, each of mass m, are arranged on the ground as shown in the figure below. A force F is applied horizontally at the bottom left cylinder at the same level as its centre of mass. If all surfaces are **smooth,** determine the minimum and maximum values of F in order for the three cylinders to remain intact.



Problem 3.2 (Ricardo)**.** A uniform inelastic string of length $L = 6.00\,\text{m}$ initially lies stretched out on a horizontal table, perpendicular to the table edge. If the coefficient of static friction between the string and the table is $\mu_s = 0.500$, what length of the string must hang over the edge before it begins to slide off the table? *The answer is not* $2.00\,m$*!*

Problem 3.3 (BAUPC 2004)**.** A point particle of mass m sits at rest on top of a frictionless, **movable** hemisphere of mass M, which rests on a frictionless table, as shown. The particle is given a tiny kick and slides down the hemisphere. Write an equation involving the angle θ (measured from the top of the hemisphere) where the particle loses contact with the hemisphere, and solve this equation for θ in the special case where $m = M$. (You need not solve this equation when $m \neq M$.)



4 Appendix

4.1 Finding Total Energy from N2L

Let's start by recalling a fundamental relationship in classical mechanics, N2L:

$$\mathbf{F} = m\mathbf{a}$$

In this case, force is a function of displacement to ensure the possibility of the force being conservative. It is crucial to recall that a potential function can only be assigned to a conservative force. Hence $\mathbf{F} = \mathbf{F}(\mathbf{r})$.

Moving on, to solve this vector differential equation, we astutely observe:

$$\mathbf{a} = \mathbf{v}\frac{d\mathbf{v}}{d\mathbf{r}} \tag{29}$$

Hence,

$$\mathbf{F} = m\mathbf{v}\frac{d\mathbf{v}}{d\mathbf{r}} \tag{30}$$

and separating the variables,

$$\mathbf{F} \cdot d\mathbf{r} = m\mathbf{v} \cdot d\mathbf{v} \tag{31}$$

Then, integrating both sides (on the left, we integrate from $\mathbf{r}_0$ to $\mathbf{r}$; on the right we integrate from $\mathbf{v}_0$ to $\mathbf{v}$),

$$\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = m\int_{\mathbf{v}_0}^{\mathbf{v}} \mathbf{v} \cdot d\mathbf{v} \tag{32}$$

$$\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v} - \frac{1}{2}m\mathbf{v}_0 \cdot \mathbf{v}_0 \tag{33}$$

Now we use the fact that $\mathbf{v} \cdot \mathbf{v} = v^2$ and $\mathbf{v}_0 \cdot \mathbf{v}_0 = v_0^2$ to simply the expressions to:

$$\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \tag{34}$$

Rearranging the terms:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \tag{35}$$

From Equation (35), we can observe that the term on the left is a constant that is dependent on the initial conditions. We then define this term to be the total energy of the system that is always a constant.

$$E \coloneqq \frac{1}{2}mv_0^2 = \text{constant} \tag{36}$$

Furthermore, we can define the second term on the right to be the potential energy:

$$U(\mathbf{r}) \coloneqq -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \tag{37}$$

Combining these with Equation (35), we can obtain the expression for total energy:

$$E = \frac{1}{2}mv^2 + U(\mathbf{r}) \tag{38}$$

4.2 Do External Forces Contribute to Potential Energy?

Let's recall the electric potential energy in Example 1.8.

For q and Q both being positive, in order for q to move from infinity to an arbitrary radial distance r away from Q, there has to be an external force that is at least the same magnitude as the electrostatic force and opposite in direction (pointing radially inwards). Let's call it $\mathbf{F}_{ext}$.

However, as we have defined potential energy to be

$$U(\mathbf{r}) := -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \tag{39}$$

where $\mathbf{F}$ is the net force on the particle, why did we simply ignore the "potential energy" due to this force?

If you are clear in your energy concept, this question may sound very silly. But it is still a valid question nonetheless. This reasoning is false because we have defined $U(\mathbf{r})$ as the potential energy of the electrostatic field produced by Q and not the total "field" on q. Thus, we naturally discard the potential contributed by the external force.

However, **sometimes we cannot discard the term contributed by the external force!** Here is an example. (If you find this confusing, you can come back to this section after learning electromagnetism.)

In the classical theory of electrodynamics, since the E-field produced by an electrostatic field is a conservative field, we can say that

$$\oint_C \mathbf{E}_{stat} \cdot d\mathbf{r} = 0 \tag{40}$$

Let's define any force per unit charge to be $\mathbf{f}$. Here $\mathbf{f} = \mathbf{E}$. In a circuit with a battery and wires connecting the two terminals of a battery, point A and B, we can compute the electromotive force (emf) by definition:

$$\varepsilon = \oint_C (\mathbf{f}_{stat} + \mathbf{f}_{non}) \cdot d\mathbf{r} = \oint_C \mathbf{f}_{stat} \cdot d\mathbf{r} + \int_B^A \mathbf{f}_{non} \cdot d\mathbf{r} = \int_B^A \mathbf{f}_{non} \cdot d\mathbf{r} \tag{41}$$

So, here you clearly cannot ignore the driving external force!

4.3 Single vs Multiple Particle COE

The idea of potential energy and how it is defined comes hand-in-hand with the notion of conservation of energy. More specifically, we define potential energy to quantitatively understand a conserved quantity known as total energy, which we empirically observed to be conserved from experiments.

Conserved quantities are the "god mods" in physics. Without them, we can practically do nothing. (Although everything in classical mechanics can, on paper be done using Newton's laws, the mathematics simply becomes too complicated to handle.) This is the precise motivation of how we define it, to be able to use these conserved quantities to solve problems we face rigorously.

Let us look at single particle COE first. Recall Equation (35). Since total energy is constant:

$$E - E = \frac{1}{2}mv^2(x_2) + U(x_2) - \frac{1}{2}mv^2(x_1) - U(x_1) = 0$$

$$\frac{1}{2}mv^2(x_2) - \frac{1}{2}mv^2(x_1) = U(x_1) - U(x_2)$$

$$\frac{1}{2}mv^2(x_2) - \frac{1}{2}mv^2(x_1) = \int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{F} \cdot d\mathbf{r}$$
$$= Work\Big|_{x_1}^{x_2}$$

Thus it is not hard to see:

$$\Delta KE = Work_{ext} \tag{42}$$

Or alternatively:

$$\Delta KE + \Delta PE = 0$$

Most of the confusion comes about when there is more than one particle involved. In this case, we have to be careful to not double count potential energy, and to do so, we have to be clear about what we define to be the system of interest.

We saw an example of double-counting in Example 1.9. Now, let's approach it quantitatively. Let's start off simple with a two-particle system.

The force on the i^{th} particle by j^{th} particle is a function of the separation vector between them:

$$\mathbf{F}_{ij} = \mathbf{F}_{ij}(\mathbf{r}_i - \mathbf{r}_j)$$

Now, due to the system of interest containing more than one particle, the external work consists of work done by an external agent and work done due to other particles in the system. Hence the work-energy theorem of particle one is:

$$W_{ext}^1 + W_{21} = \Delta T_1$$

Hence the work-energy theorem of particle two is:

$$W_{ext}^2 + W_{12} = \Delta T_2$$

When combining these two expressions:

$$(W_{ext}^1 + W_{ext}^2) + (W_{21} + W_{12}) = \Delta(T_1 + T_2)$$

$$W_{ext} + \int_{\mathbf{r}_{10}}^{\mathbf{r}_{11}} \mathbf{F}_{12} \cdot d\mathbf{r}_1 + \int_{\mathbf{r}_{20}}^{\mathbf{r}_{21}} \mathbf{F}_{21} \cdot d\mathbf{r}_2 = \Delta T_{tot}$$

Here is where the magic happens. First, we recall that by Newton's Third Law, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. Hence we can rewrite our expression to:

$$W_{ext} + \int_{\mathbf{r}_{10}}^{\mathbf{r}_{11}} \mathbf{F}_{12} \cdot d\mathbf{r}_1 - \int_{\mathbf{r}_{20}}^{\mathbf{r}_{21}} \mathbf{F}_{12} \cdot d\mathbf{r}_2 = \Delta T_{tot} \tag{43}$$

Next, to convince you that what I am about to do is true, let's consider more fundamental mathematics. Let's define $z = x + y$. From the product rule, $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ and from $z = x + y$, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} = 1$. Thus $dz = dx + dy$ and $d(x + y) = dx + dy$.

Going back to our example, we have $\mathbf{F}_{12} \cdot d\mathbf{r}_1 - \mathbf{F}_{12} \cdot d\mathbf{r}_2$. We can the factorise the force out and obtain:

$$\mathbf{F}_{12} \cdot (d\mathbf{r}_1 - d\mathbf{r}_2)$$

From what we concluded before, albeit not very rigorous mathematically, but enough for our purpose, we can then rewrite this as:

$$\mathbf{F}_{12} \cdot d(\mathbf{r}_1 - \mathbf{r}_2)$$

Then we can express:

$$W_{ext} + \int_{\mathbf{r}_{10}-\mathbf{r}_{20}}^{\mathbf{r}_{11}-\mathbf{r}_{21}} \mathbf{F}_{12} \cdot (d\mathbf{r}_1 - d\mathbf{r}_2) = \Delta T_{tot}$$

Now if we define the potential energy between particle one and particle two to be:

$$U_{12} = -\int_{\mathbf{r}_{10}-\mathbf{r}_{20}}^{\mathbf{r}_{11}-\mathbf{r}_{21}} \mathbf{F}_{12} \cdot (d\mathbf{r}_1 - d\mathbf{r}_2)$$

We can rewrite everything in a very simple fashion:

$$W_{ext} = \Delta T_{tot} + \Delta U_{12}$$

To make things more clear what U_{12} is, let us define $\mathbf{r}' = \mathbf{r}_1 - \mathbf{r}_2$, then it is clear that U_{12} only depends on the relative separation of the particles. So we can hold one particle fixed and bring the other from the origin to $\mathbf{r}'$

$$U_{12}(\mathbf{r}_1 - \mathbf{r}_2) = -\int_{\mathbf{r}'_0}^{\mathbf{r}'_1} \mathbf{F}_{12} \cdot d\mathbf{r}'$$

This also explains the spring example we had.

Now finally we can move on to a multiple-particle system. All the precious work we have done makes the transition extremely effortless. We simply claim that for an N particle system, the total internal potential energy associated with the interactions between pairs of particles is:

$$U_{int} = \sum_{i<j}^{N} U_{ij}$$

Thus, the work-energy theorem for a multi-particle system with only conservative force is:

$$Work_{ext} = \Delta T_{tot} + \Delta U_{int}$$

Alternatively,

$$\Delta T_{tot} + \Delta U_{ext} + \Delta U_{int} = 0 \tag{44}$$

Now we will be presenting a crucial example that wil be enlightening.

Example 4.1. A satellite is initially in orbit around the Earth at some height h, but it activates its propulsion rockets and eventually transits to some higher orbit $h' > h$. You are not given the intermediate trajectory of the satellite. Assume all orbits are circular. Did the force due to the rockets do work on the satellite? If so, is it positive or negative? Did the Earth's gravity do work on the satellite? If so, is it positive or negative?

Solution. Firstly, the force on the rocket obviously did positive work on the satellite because it is propelling it. Secondly, the gravitational force did negative work on the satellite because the satellite moved in the opposite direction as the force.

Let us first consider the satellite as the sole object in our system. $Work_{ext}$ in this case consists of $Work_{grav} + Work_{rocket}$ and there is no change in internal energy. Hence our conservation of energy statement becomes:

$$Work_{grav} + Work_{rocket} = \Delta T$$

$$\Delta T + \Delta U_{grav} + \Delta U_{rocket} = 0$$

Next, let us consider the satellite and rocket system. $Work_{ext}$ in this case consists of $Work_{grav}$ only as our $\Delta U_{int} = \Delta U_{rocket}$. This can be a change in the chemical potential energy of the rocket. Hence our conservation of energy statement becomes:

$$Work_{grav} = \Delta T + \Delta U_{rocket}$$

$$\Delta T + \Delta U_{grav} + \Delta U_{rocket} = 0$$

Lastly, let us consider the satellite-rocket-planet system. $Work_{ext}$ in this case is zero as there is no external force in our system. $\Delta U_{int} = \Delta U_{rocket} + \Delta U_{grav}$. Hence our conservation of energy statement becomes:

$$\Delta T + \Delta U_{grav} + \Delta U_{rocket} = 0$$

As you can see, no matter how we define our system, as long as we are meticulous with our work, we will arrive at the same answer regardless.